Translational bounds for factorial n and the factorial polynomial

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During the period 1729–1826 Bernoulli, Euler, Goldbach and Legendre developed expressions for defining and evaluating $n!$ and the related gamma function. Expressions related to $n!$ and the gamma function are a common feature in computer science and engineering applications. In the modern computer age we live in now, two common tests to determine improved power of new computers are finding new large prime numbers and calculating $n!$ for larger values of n. Therefore it is useful to know how quickly $n!$ grows and what tight bounds exist.

Polynomial bounds for $n!$ are typically of degree n. We give below a family of bounds of degree $n - k$. For example

$$6\left(\frac{n+4}{2}\right)^{n-3}$$

is a much better bound for $n!$ than the commonly used

$$\left(\frac{n+1}{2}\right)^{n}$$

The other theme is to give bounds for $(n + k)!$ (translational bounds) to complement the known bounds for $(kn)!$ (multiplicative bounds). The method used is to obtain a translational bound for the factorial polynomial in order to create a translational bound for $n!$ as a corollary.

The result for $(n + k)!$ also yields as corollaries, bounds for $(2n)!$, $(3n)!$ and the general $(kn)!$, which confirm the corresponding results in Mahmood, Edwards and Singh (1999). The $AM \le GM$ inequality is used to obtain Theorem 4. The sequence

$$\left\{\left(1+\frac{1}{n}\right)^{n}\right\}$$

is of course an increasing sequence converging to e. For results connecting this sequence and $n!$ and the *AM-GM* inequality, see Boskoff and Suceava (2006), and Mahmood and Edwards (1999).

Background

The famous approximation of $n!$ due to Stirling is that $n! \approx S_n$, where

$$S_n = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

It is also known (Feller, 1968) that $Sn \le n! \le Sn(e^{1/12n})$.
A simpler (but wider) set of bounds are

$$\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n}{2}\right)^n$$

Mathematicians have long been interested in polynomial bounds for $n!$ with the most famous being the straightforward

$$n! \le \left(\frac{n+1}{2}\right)^n$$

Since $e \approx 2.71828 > 2$,

$$n! \approx S_n < \left(\frac{n}{2}\right)^n$$

for large n and also $n! \approx S_n > \dfrac{n^n}{3^n}$ so $\left(\dfrac{n}{3}\right)^n < n! < \left(\dfrac{n}{2}\right)^n$ for large n.

In fact, for large n,

$$\left(\frac{n}{p}\right)^n < n! < \left(\frac{n}{q}\right)^n$$

whenever $0 < q < e \le p$. Mahmood, Edwards and Singh (1999) showed that

$$(kn)! \le [(k-1)n]!\left(\frac{(2k-1)n+1}{2}\right)^n \le \left(\frac{1+kn}{2}\right)^{kn}$$

and so for example

$$(3n)! \le \left(\frac{n+1}{2}\right)^n\left(\frac{3n+1}{2}\right)^2\left(\frac{5n+1}{2}\right)^n$$

Edwards and Mahmood (2001) demonstrated that for each integer $a \ge 0$, there exists n_a such that

$$n! < \left(\frac{n+1-a}{2}\right)^n$$

for all $n > n_a$. Thus

$$n! < \left(\frac{n+1}{2}\right)^n$$

for all $n > 10$ and

$$n! < \left(\frac{n-10}{2}\right)^n$$

for all $n > 45$. Mahmood and Edwards (2001) also included the general result that for each real number b, $0 < b < e$, there exists n_b such that

$$n! < \left(\frac{n}{b}\right)^n, \ \forall n > n_b$$

Thus $n! < \left(\frac{n}{2.7}\right)^n$ for all $n > 611$ and even $n! < \left(\frac{n}{2.718}\right)^n$ for all $n > 62\,083$.

The polynomial bounds mentioned above all are of degree n and bounds were given for $(kn)!$. In the next section, polynomial bounds of degree less than n and bounds for $(n + k)!$ are given. We view these as *translational* bounds which contrast with the *multiplicative* bounds for $(kn)!$.

Main results

The factorial polynomial $x^{(n)}$ is defined by $x^{(0)} = x(x-1)(x-2)\ldots(x-n+1)$. For other results concerning the factorial polynomial, see Mahmood, Edwards, and Singh (1999).

Theorem 1

$x^{(n)} \le \left(x - \frac{n-1}{2}\right)^n$ (Mahmood, Edwards, & Singh, 1999)

Lemma 2

(i) $(x+k)^{(n+k)} = (x+k)^{(n)}(x+k-n)^{(k)}$

(ii) $(x+k)^{(n+k)} = (x+k)^{(k)}(x)^{(n)}$

Proof

$(x+k)^{(n+k)}$

$=[(x+k)(x+k-1)\ldots(x+k-n+1)][(x+k-n)(x+k-n)(x+k-n-1)\ldots(x+k-n-k+1)]$

$=[(x+k)\ldots(x+k-n+1)][(x+k-n)\ldots(x+k-n-k+1)]$

$=(x+k)^{(n)}(x+k-n)^{(k)}$

Also

$(x+k)^{(n+k)}$

$=[(x+k)(x+k-1)\ldots(x+k-k+1)][(x+k-k)(x+k-k-1)\ldots(x+k-k-k-n+1)]$

$=[(x+k)\ldots(x+k-k+1)][x\ldots(x-n+1)]$

$=(x+k)^{(k)}(x)^{(n)}$

Theorem 3

For any non-negative integer k,

$$\left(x+k\right)^{(n+k)} \le \left(x+k-n\right)^{(k)}\left(\frac{2x+2k-n+1}{2}\right)^{n}$$

Proof

Replace x by $x + k$ in Theorem 1 and use Lemma 2(i).

Remark 1

Even though Theorem 1 was just used to prove Theorem 3, Theorem 1 is a special case of Theorem 3 where $k = 0$. Thus the two theorems are equivalent in this sense.

Remark 2

Theorem 3 above reduces to the following important factorial result when $x = n$ and noting that $m^{(m)} = m!$.

Theorem 4

For any non-negative integer k,

$$\left(n+k\right)! \le k!\left(\frac{n+2k+1}{2}\right)^{n}$$

Corollary 5

For any non-negative integer $k \le n$,

$$n! \le k!\left(\frac{n+k+1}{2}\right)^{n-k}$$

Proof

Just replace n by $n - k$ in Theorem 4.

The next four corollaries follow from Corollary 5. The first is the standard bound for $n!$ and the latter three are improved bounds.

Corollary 6

$$n! \le 0!\left(\frac{n+1}{2}\right)^{n}$$

Corollary 7

$$n! \le 1!\left(\frac{n+2}{2}\right)^{n-1}, \quad n \ge 1$$

Corollary 8

$$n! \le 2!\left(\frac{n+3}{2}\right)^{n-2}, \quad n \ge 2$$

Corollary 9

$$n! \le 3!\left(\frac{n+4}{2}\right)^{n-3}, \; n \ge 3$$

It is clear, for example, that

$$6\left(\frac{n+4}{2}\right)^{n-3}$$

is a much better bound for $n!$ than

$$\left(\frac{n+1}{2}\right)^{n}$$

Applications

Suppose we know 10! but we do not know 15!. Then using $k = 10$ and $n = 5$ in Theorem 4:

$$15! \le 10!\left(\frac{5+20+1}{2}\right)^{5}$$

i.e., $15! \le 10! \le (13)^5$ which is a much better bound than

$$15! \le \left(\frac{15+1}{2}\right)^{15}$$

in fact, about 26 times better (smaller).

Boskoff and Suceava (2006) showed that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}$$

for positive integers m and n.
We rewrite the above as

$$(n+k)! < (n+k)^{n+k} \frac{k!}{k^k} \cdot \frac{n!}{n^n}$$

Using $k = 10$ and $n = 5$ above yields

$$15! < (15)^{15} \frac{10!}{(10)^{10}} \cdot \frac{5!}{(5)^5}$$

which is not sharper (or simpler) than $15! \le 10!(13)^5$. In fact our bound is about 4.5 times better. A further application of Theorem 4 is to prove the following three results which were given in Mahmood, Edwards, and Singh (1999).

Corollary 10

$$(2n)! \le n!\left(\frac{3n+1}{2}\right)^n \le \left(\frac{n+1}{2}\right)^n\left(\frac{3n+1}{2}\right)^n$$

Proof

Put $k = n$ in Theorem 4 and note that $n! \le \left(\frac{n+1}{2}\right)^n$.

Corollary 11

$$(3n)! \le 2n!\left(\frac{5n+1}{2}\right)^n \le \left(\frac{n+1}{2}\right)^n\left(\frac{3n+1}{2}\right)^n\left(\frac{5n+1}{2}\right)^n$$

Proof

Put $k = 2n$ in Theorem 4 and use Corollary 10.

Corollary 12

$$(kn)! \le \left[(k-1)n\right]!\left(\frac{(2k-1)n+1}{2}\right)^n$$

Proof

Substitute $(k-1)n$ for k in Theorem 4.

Conclusion

Polynomial bounds of degree less than n were given for $n!$. We derived extended bounds for the bounds of $n!$ to $(n+k)!$ and $x^{(n)}$ to $x^{(n+k)}$. We termed them as translational bounds and the bounds were demonstrated for $(n+k)!$. It was shown that the bound for $(n+k)!$ is much better than the translational bound of Boskoff and Suceava (2006). The new translational bounds confirm the bounds on factorial n of Mahmood, Edwards, and Singh (1999) and Mahmood and Edwards (1999), as well as the bounds for $(2n)!$, $(3n)!$ and the general $(kn)!$ of Mahmood, Edwards, and Singh (1999).

References

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